Surface waves determination from pressure measurements at the bottom

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ABSTRACT. – Measuring a free surface wave motion without intrusive probes is of most importance for practical applications, and one way to achieve this is to use pressure gauges at the bottom. From these pressure measurements, the question is how one can reconstruct the free surface. In this work, we present a method of surface recovery from bottom pressure, in the context of fully nonlinear waves. The problem is solved analytically for steady flows.

Key-words: surface gravity waves, pressure measurement, surface recovery

Détermination des vagues à partir de jauges de pression

RÉSUMÉ. – Dans le cadre des ondes de gravité (vagues), on présente une méthode pour reconstruire la surface libre à partir de mesures de pression sur le fond. Le problème est résolu analytiquement pour les écoulements permanents.

Mots-clés : onde de gravité, mesure de pression, reconstruction de surface

I. INTRODUCTION

Within the framework of incompressible, homogeneous and inviscid flow modelling of water waves, the pressure plays an important role in establishing various qualitative properties of traveling waves. The pressure is also essential in quantitative studies because often in practice the elevation of a surface water-wave is determined from pressure data obtained at the bottom, see, e.g., Tsai *et al.* (2005). Indeed, measuring a free surface wave motion without intrusive probes is of most importance for practical applications, and one way to achieve this is to use pressure gauges at the bottom. From these pressure measurements, the question is how one can reconstruct the free surface.

A standard approach consists in assuming that the hydrostatic approximation is sufficiently accurate. However, already for waves of moderate amplitude, prediction errors exceeding 15% frequently occur (Bishop & Donelan 1987). These inaccuracies are due to the fact that the hydrostatic approximation ignores even linear wave effects. Within the linear regime of water waves of small amplitude in finite depth, one can derive a better approximation (Escher & Schlurmann 2008), but its benefit is offset by the fact that, for waves of moderate amplitude, it often overestimates the wave height by more than 10% (Tsai *et al.* 2005). These considerations motivated the quest for a reconstruction formula that accounts for nonlinear effects and that is thus applicable to waves of moderate and large amplitude. Nonlinear nonlocal equations relating the dynamic pressure on the bed and the wave profile were obtained without approximation from the governing equations (Constantin 2012; Deconinck *et al.* 2012). The entangled character of these equations is a considerable impediment in the pursuit of an in-depth analysis and for practical applications.

Very recently, new exact tractable relations have been obtained (Clamond 2013; Clamond & Constantin 2013), allowing mathematical analysis and a straightforward numerical procedure for deriving the free surface from the pressure at the bed. The present paper is a presentation of these relations and an illustration of their simplicity.

II. HYPOTHESIS & NOTATIONS

In a frame of reference moving at the constant wave speed c, the flow beneath a traveling wave reduces to a steady motion with respect to the moving coordinate system. Let (x, y) be a Cartesian coordinate system moving with the wave, x being the horizontal coordinate and y the upward vertical coordinate. Let $\{u(x,y), v(x,y)\}$ be the velocity field in the moving frame. The wave is $(2\neq/k)$-periodic in the x-direction, and we denote by $y=-d$, $y=\eta(x)$ and $y=0$ the equations of the horizontal bottom, of the free surface and of the mean water level, respectively. The latter equation expresses the fact that $<\eta>=0$ for the smooth $(2\neq/k)$-periodic wave profile η, where $<\cdot>$ is the Eulerian average operator over one wavelength,

$$<\eta>\equiv \frac{k}{2\pi}\int_{-\pi/k}^{\pi/k}\eta(x)\,\mathrm{d}x=0. \tag{1}$$

The flow is governed by the balance between the restoring gravity force and the inertia of the system. With constant density $\rho>0$, the equation of mass conservation and Euler's equations are respectively

$$u_x+v_y=0, \tag{2}$$

$$uu_x+vu_y=-P_x/\rho, \tag{3}$$

$$uv_x+vv_y=-g-P_y/\rho, \tag{4}$$

where $P(x,y)$ denotes the pressure. We are investigating gravity water waves, so that the assumption of a homogeneous non-viscous fluid is appropriate. The effect of surface tension being negligible, on the free surface we must have $P = P_{\mathrm{atm}}$ on $y = \eta(x)$, where P_{atm} is the (constant) atmospheric pressure. The fact that the free surface and the rigid bed are interfaces is captured by the kinematic boundary conditions $v = u\eta_x$ on $y = \eta(x)$ and $v = 0$ on $y = -d$, respectively, while the irrotational character of the flow is ensured by requiring $u_y = v_x$.

Let φ and ψ be the velocity potential and the stream function, respectively, such that $u = \varphi_x = \psi_y$ and $v = \varphi_y = -\psi_x$. It is convenient to introduce the complex potential and the complex velocity $f = \varphi + \mathrm{i}\psi$ and $w = u - \mathrm{i}v$, respectively, that are holomorphic functions of the complex variable $z = x + \mathrm{i}y$, with $f = f(z)$ and $w = \mathrm{d}f / \mathrm{d}z$. The equation of mass conservation for a homogeneous fluid and the irrotational character of the flow are identically fulfilled with the requirement that f and w are holomorphic functions throughout the fluid domain. The Euler equation can be expressed by means of the Bernoulli condition

$$2p + 2gy + u^2 + v^2 = B, \qquad x \in R, -d \leqslant y \leqslant \eta(x), \quad (5)$$

for some Bernoulli constant B, where $p = p(x,y)$ is the pressure divided by the density. From (1) and (5), we get

$$B = < u_{\mathrm{s}}^2 + v_{\mathrm{s}}^2 > = < u_{\mathrm{b}}^2 >, \quad (6)$$

where u_{s} and v_{s} denote the restrictions of u and v to the free surface, respectively, while u_{b} denotes the restrictions of u at the bed. (The second equality in (6) derives from the irrotationality.) The relations (5) and (6) yield

$$< p_{\mathrm{b}} > = gd, \quad (7)$$

where $p_{\mathrm{b}}(x) = p(x,-d)$ is the normalized relative pressure at the bed.

Finally, we define the wave phase velocity c such that

$$c = - < u_{\mathrm{b}} >, \quad (8)$$

so that the wave travels with phase speed c in the frame of reference where the mean horizontal velocity is zero at the bed, and where $c > 0$ if the wave travels toward the increasing x-direction.

III. EQUATIONS FOR THE SURFACE RECOVERY

Instead of dealing with the complex potential f or with the complex velocity w, it is advantageous to use the holomorphic function w^2. Indeed, the function $\mathcal{P}$ defined by

$$\mathcal{P}(z) \equiv 12B + gd - 12w^2(z) = 12B + gd - 12(u^2 - v^2) + \mathrm{i}uv, \quad (9)$$

is holomorphic in the fluid domain and its restriction to the flat bed $y = -d$ has zero imaginary part and real part p_{b}, i.e., $p_{\mathrm{b}}(x) = \mathcal{P}(x - \mathrm{i}d) = gd + 1/2(B - u_{\mathrm{b}}^2)$. Thus p_{b} determines $\mathcal{P}$ uniquely throughout the fluid domain, i.e., $\mathcal{P}(z) = p_{\mathrm{b}}(x + \mathrm{i}y + \mathrm{i}d)$. Note that p coincides with the real function $Re\{\mathcal{P}\}$ only on $y = -d$ because p is not a harmonic function in the fluid domain.

Using the surface impermeability and (5) on the free surface $y = \eta(x)$ where $p = 0$, we have

$$(u_{\mathrm{s}} - \mathrm{i}v_{\mathrm{s}})^2 = (1 - \mathrm{i}\eta_x)^2 u_{\mathrm{s}}^2 = (1 + \eta_x^2) u_{\mathrm{s}}^2 (1 - \mathrm{i}\eta_x)/(1 + \mathrm{i}\eta_x)$$

$$= (u_{\mathrm{s}}^2 + v_{\mathrm{s}}^2)(1 - \mathrm{i}\eta_x)/(1 + \mathrm{i}\eta_x) = (B - 2g\eta)(1 - \mathrm{i}\eta_x)/(1 + \mathrm{i}\eta_x). \quad (10)$$

Multiplying this relation by $(1 + \mathrm{i}\eta_x)$ and using (9), we obtain at once

$$g\eta(1 - \mathrm{i}\eta_x) + \mathrm{i}B\eta_x = [\mathcal{P}(x + \mathrm{i}\eta) - gd](1 + \mathrm{i}\eta_x). \quad (11)$$

The real and imaginary parts of (11) give two equations for η:

$$g\eta = Re\{\mathcal{P}_{\mathrm{s}}\} - gd - \eta_x Im\{\mathcal{P}_{\mathrm{s}}\}, \quad (12)$$

$$(B - g\eta)\eta_x = [Re\{\mathcal{P}_{\mathrm{s}}\} - gd]\eta_x + Im\{\mathcal{P}_{\mathrm{s}}\}, \quad (13)$$

where, as above, the subscript 's' denotes the evaluation at the free surface $y = \eta(x)$. Using (9), we can see that equations (13) are precisely

$$g\eta = \frac{1}{2}[B - u_{\mathrm{s}}^2 - v_{\mathrm{s}}^2], \qquad (B - g\eta)\eta_x = \frac{1}{2}[B + u_{\mathrm{s}}^2 + v_{\mathrm{s}}^2]\eta_x. \quad (14)$$

Thus, both are ensured by the validity of the Bernoulli condition (5) on the free surface. Since $\eta_x \neq 0$ between consecutive crests and troughs, not only does (14 *a*) imply (14 *b*), but also (14 *b*) ensures the validity of (14 *a*) between consecutive crests and troughs, and by continuous extension everywhere.

For the recovery of the surface wave profile η, given the function p_{b}, one can proceed as follows. For periodic waves, the pressure p at the bed $y = -d$ can be approximated, e.g., by a N-th order Fourier polynomial and the function $\mathcal{P}$ is obtained at once, i.e.,

$$p_{\mathrm{b}}(x) \approx \sum_{n=-N}^{N} p_n \exp(\mathrm{i}nkx)$$
$$\Rightarrow \qquad \mathcal{P}(z) \approx \sum_{n=-N}^{N} p_n \exp(\mathrm{i}nk[x + \mathrm{i}(y + d)]), \quad (15)$$

with $p_{-n} = \overline{p}_n$ since p_{b} is real, and $p_0 = gd$. Note that (15) is not the only possible approximation and, e.g., elliptic functions could also be used, specially in shallow water.

The wave amplitude η_0 is obtained evaluating (13 *a*) at the wave crest located at $x = 0$ (where $\eta_x = 0$ and $\eta = \eta_0$) leading to the implicit equation

$$\eta_0 = Re\{\mathcal{P}(\mathrm{i}\eta_0)\}/g - d. \quad (16)$$

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The crest height η_0 is obtained as the unique solution to (16), as proven by Clamond & Constantin 2013) and where numerical examples are provided. With η_0 determined by solving (16) iteratively, η is subsequently obtained re-expressing (13 *b*) as the ordinary differential equation

$$\eta_x = Im\{\mathcal{P}_s\}/\left[B - g\eta - Re\{\mathcal{P}_s\} + gd\right], \qquad (17)$$

with initial data $\eta(0) = \eta_0$. The right-hand side of (17) being smooth, the solution can be obtained by a standard iterative procedure (Clamond & Constantin 2013). It is however possible to derive a local simpler expression which does not require the resolution of a differential equation.

Note that the surface reconstruction procedure described here is valid for all waves, except perhaps for the highest ones with an angular crest and for waves with different crests (Clamond & Constantin 2013). Note also that the inclusion of surface tensions is straightforward, but this generalisation has little practical interest.

IV. SIMPLIFIED EQUATIONS FOR THE RECOVERY

Let be yet another holomorphic function Q such that

$$Q(z) \equiv \int_{z_0}^{z}\left[\mathcal{P}(z') - gd\right]\mathrm{d}z' = \int_{z_0}^{z} 12\left[B - w(z')^2\right]\mathrm{d}z', \qquad (18)$$

where z_0 is an arbitrary constant. Taking z_0 at the origin of the free surface — i.e., $x = 0$, $y = \eta_0$ thence $z_0 = \mathrm{i}\eta_0$ — and choosing the surface as integration path, the definition (18) yields

$$Q_s(x) = \int_0^x \left[\mathcal{P}(x' + \mathrm{i}\eta(x')) - gd\right][1 + \mathrm{i}\eta_x(x')]\mathrm{d}x', \qquad (19)$$

thence, substituting (11), after some elementary algebra

$$Q_s(x) = \int_0^x g\eta(x')\mathrm{d}x' + \mathrm{i}\left[\eta(x) - \eta_0\right]\left[B - 12g\eta_0 - 12g\eta(x)\right]. \qquad (20)$$

The imaginary part of this relation yields an implicit equation for η:

$$\kappa\eta = 1 - \sqrt{(1 - \kappa\eta_0)^2 - (2\kappa / B)Im\{Q_s\}}, \qquad (21)$$

where $\kappa \equiv g / B$ is a parameter introduced for convenience. The relation (21) is algebraic (i.e., neither differential nor integral) and local. η can be obtained via functional iterations once O is known. Once the pressure at the bottom is known, it is trivial to obtain $\mathcal{P}$ as indicated above and, subsequently, O is easily obtained too, for example from the approximation (15)

$$Q(z) \approx \sum_{n \neq 0} \frac{p_n}{\mathrm{i}nk}\left[\exp(\mathrm{i}nkz) - \exp(-nk\eta_0)\right]\exp(-nkd). \qquad (22)$$

The remaining open question is whether or not the iterations of (21) converge. The convergence does occur for all waves, except perhaps for the highest waves (Clamond 2013).

V. CONCLUSION

We derived some exact relations for the reconstruction of the free surface wave profile from pressure measurements at the bed. These relations allow an easy and fast reconstruction, which is interesting for practical applications. Efficient algorithms and easy proofs then follow at once.

For practical applications, the method has to be adapted to take into account that the flow are never: exactly permanent, irrotational, etc. The recovery procedure described here is nevertheless promising. Further details and practical implementations will be discussed in future works.

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